Or, in the original variable

$$x = I\sin\varphi_1 = \left(J - \frac{3}{2}\varepsilon J^3\right)\sin\gamma_1 - \frac{\varepsilon}{8}[2J^3\sin 3\gamma_1 + \mu\sin(3\gamma_1 - \gamma_2)]$$

In the problem considered we could manage without the introduction of adjoint variables, since the original system can be written immediately in Hamiltonian form.

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ON THE CORNER POINTS OF THE BOUNDARIES OF REGIONS OF ATTAINABILITY*

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Some properties of the boundaries of the regions of attainability of linear unsteady systems with a single control (perturbation) function that has values on the segment are studied. It is established that for fairly small time intervals the boundary of the attainment region has conical corner points, edges and faces. The conditions for which the distance of the conical corner points from the origin of coordinates is a maximum are established.

1. The regions of attainability of controllable (perturbable) systems were studied in /1–8/ and elsewhere. The interest in investigating such regions is connected, for instance, with the Bulgakov problem of the accumulation of perturbations /9/. The method of attainment regions was used when constructing optimal-control theory /2–4/ and the theory of games /5, 6/. Their determination is important in many applications. In /10, 11/ it was proposed to evaluate these regions by using ellipsoids. In the present paper certain statements are proved on the presence or absence of corner points ("tapered" points /5/) on the boundary of the attainment regions, on the properties of such points, and on their extremal properties. The presence of corner points is indicative of the limited nature of the approach in which the boundaries are approximated by smooth surfaces. The question of the extremal properties of boundary points arises when determining the control that removes the system farthest away from the origin of coordinates.

Consider a system defined by the linear matrix differential

$$dx/dt = A(t)x + b(t)u, \quad |u(t)| \leqslant 1 \tag{1.1}$$

where $x, A(t)$ and $b(t)$ are matrices of the type $(n \times 1), (n \times n)$ and $(n \times 1)$, respectively, and $u(t)$ is the control (perturbing) function bounded in absolute value and piecewise continuous; the set of such functions will be denoted by Ω. We will assume that the matrix elements $A(t)$ and $b(t)$ have continuous derivatives up to the $(n-1)$-th order for all t.

The solution of system (1.1) at the instant $t = T$ when $x(t_0) = 0$ is described by the integral

$$x(T) = \int_{t_0}^{T} \theta(T)\theta^{-1}(\tau)\, b(\tau)\, u(\tau)\, d\tau \tag{1.2}$$

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where $\theta(t)$ is the fundamental matrix of solutions of the uniform $(u(t) \equiv 0)$ system (1.1).

Consider, in the phase space X, the attainment region

$$Q(t_0, T) = \{x(T):\ u(t) \in \Omega\}$$

which represents the manifold of those and only those points of space X which can be reached by system (1.1) during the time $T - t_0$, moving from the origin of coordinates under the control $u(t) \in \Omega$. The manifold $Q(t_0, T)$ is convex, closed and symmetric about the origin of coordinates /1–8/.

We denote by $\Pi(\eta)$ the supporting hypersurface of the manifold $Q(t_0, T)$ orthogonal to the vector η (Fig.1), and by $d(\eta)$ the distance between the origin of coordinates and the plane $\Pi(\eta)$ (and the plane $\Pi(-\eta)$). We assume the vector to be everywhere unitary, i.e. $\eta \in S$, where $S = \{\eta: \|\eta\| = 1\}$. Those points and only those points x belong to the manifold $Q(t_0, T)$ whose coordinates satisfy the inequalities

$$|\eta x| \leqslant d(\eta), \quad \forall \eta \in S \tag{1.3}$$

where η is the row-matrix $(1 \times n)$. For the distance $d(\eta)$ we have the expression /8/

$$d(\eta) = \max_{x(T) \in Q(t_0, T)} (\eta x(T)) = \max_{u(\tau) \in \Omega} \int_{t_0}^{T} \eta\theta(T)\theta^{-1}(\tau) b(\tau) u(\tau)\, d\tau = \operatorname{sgn}(T - t_0) \int_{t_0}^{T} |\eta\theta(T)\theta^{-1}(\tau) b(\tau)|\, d\tau \tag{1.4}$$

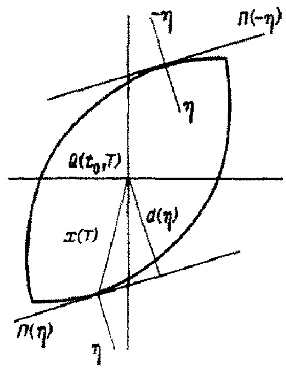

Fig.1

The control $u(\tau)$ that gives a maximum value to the functional in (1.4) has the form

$$u(\tau) = u(\eta, \tau) = \operatorname{sgn}[\eta\theta(T)\theta^{-1}(\tau) b(\tau)(T - t_0)] \tag{1.5}$$

We will assume that on any segment of the axis $-\infty < \tau < \infty$, we have

$$\psi(\eta, T, \tau) = \eta\theta(T)\theta^{-1}(\tau) b(\tau) \not\equiv 0, \quad \forall \eta \in S, \quad \forall T \tag{1.6}$$

It was shown in /4/ that inequality (1.6) holds, if for all t

$$\operatorname{rank} \| l_1(t), \ldots, l_n(t) \| = n \tag{1.7}$$

$$\left(l_1(t) = b(t),\ l_k(t) = A(t) l_{k-1}(t) - \frac{dl_{k-1}}{dt},\ k = 2, \ldots, n\right)$$

Under condition (1.7) the function $\psi(\eta, T, \tau)$ over any segment has only a finite number of zeros /4/.

This condition is used below to prove a number of statements. For systems with constant matrices A and b the inequality (1.6) is equivalent to the Kalman controllability property /12/.

Under condition (1.6) the maximizing control (1.5) is unique. Hence the boundary point $x(T)$ at which the hyperplane $\Pi(\eta)$ touches the manifold $Q(t_0, T)$ is also unique (this implies the strict convexity of $Q(t_0, T)$) and is given by the formula

$$x(T) = \int_{t_0}^{T} \theta(T)\theta^{-1}(\tau) b(\tau) \operatorname{sgn}[\eta\theta(T)\theta^{-1}(\tau) b(\tau)(T - t_0)]\, d\tau \tag{1.8}$$

which is the parametric equation (parameter $\eta \in S$) of the boundary $\Gamma(t_0, T)$ of the region of $Q(t_0, T)$. It can be considered as the mapping of the sphere S onto the boundary

$$\Gamma(t_0, T) = \Phi(S, t_0, T) \tag{1.9}$$

The strict convexist of the manifold $Q(t_0, T)$ for any t_0, T is equivalent to the uniqueness of mapping (1.9). If the boundary $\Gamma(t_0, T)$ was everywhere smooth, i.e. at each point $x \in \Gamma(t_0, T)$ a unique support (that is also tangent) hyperplane $\Pi(\eta)$ of manifold $Q(t_0, T)$ existed, then the mapping

$$S = \Phi^{-1}[\Gamma(t_0, T)] \tag{1.10}$$

which is the inverse of mapping (1.9), would also be unique. But the boundary $\Gamma(t_0, T)$ is generally not everywhere smooth (which makes mapping (1.10) nonunique). This circumstance is the subject of further investigation.

2. The corner points of the boundary.

We introduce the following definitions.

Definition 1. We shall call the point $x \in \Gamma(t_0, T)$ the corner point of the boundary, if that point is contained by at least two different ($\eta^{(1)} \neq \eta^{(2)}$) supporting hyperplanes $\Pi(\eta^{(1)})$ and $\Pi(\eta^{(2)})$ of the manifold $Q(t_0, T)$.

Let $x \in \Gamma(t_0, T)$ be a corner point and $x \in \Pi(\eta^{(1)}), \Pi(\eta^{(2)})$. Then, owing to the uniqueness of the maximizing functions (1.5)

$$u(\eta^{(1)}, \tau) = u(\eta^{(2)}, \tau), \quad \forall \tau \in (t_0, T) \tag{2.1}$$

i.e.

$$\operatorname{sgn}[\eta^{(1)}\theta(T)\theta^{-1}(\tau)b(\tau)] = \operatorname{sgn}[\eta^{(2)}\theta(T)\theta^{-1}(\tau)b(\tau)] \quad \forall \tau \in (t_0, T) \tag{2.2}$$

We will rewrite Eq. (2.2) for constant matrices A and b

$$\operatorname{sgn}(\eta^{(1)}e^{A\xi}b) = \operatorname{sgn}(\eta^{(2)}e^{A\xi}b), \quad \forall \xi \in (0, T - t_0) \tag{2.3}$$

Let us assume now that the converse is true, i.e. that two different vectors $\eta^{(1)}, \eta^{(2)} \in S$ exist such that identity (2.2) holds, which means that (2.1) also holds. Then, two supporting hyperplanes $\Pi(\eta^{(1)})$ and $\Pi(\eta^{(2)})$ exist that contain one and the same point $x \in \Gamma(t_0, T)$.

Thus the following statement holds.

Lemma 1. For a corner point to exist on the boundary $\Gamma(t_0, T)$ it is necessary and sufficient that at least two different vectors $\eta^{(1)}, \eta^{(2)} \in S$ exist, for which identity (2.2) holds.

Lemma 2. For the point $x \in \Gamma(t_0, T)$ corresponding to the vector $\eta^{(1)} \in S$ to be a smooth point of the boundary $\Gamma(t_0, T)$ (not a corner point) it is necessary and sufficient that a neighbourhood $S^{(1)} \in S$ of the vector $\eta^{(1)}$ exists such that for any vector $\eta^{(2)} \in S^{(1)}$ identity (2.2) does not hold.

By analyzing the conditions of identities (2.2) (or (2.3)) it is possible to obtain the conditions under which the boundary of the attainment region has, or has not, corner points.

Examples. Consider the following system of differential equations

$$x_1^{\cdot} = x_2, \quad (m - \mu t)x_2^{\cdot} = u \quad (m, \mu = \text{const} > 0, \ 0 \leqslant t < m/\mu) \tag{2.4}$$

On certain assumptions system (2.4) defines a controllable (perturbed) motion of a solid of variable mass around its axis of symmetry.

For second-order systems the vector $\eta(\eta_1, \eta_2)$ may be represented in the form

$$\eta_1 = \cos\varphi, \ \eta_2 = \sin\varphi \quad (0 \leqslant \varphi \leqslant 2\pi) \tag{2.5}$$

Then

$$\eta\theta(T)\theta^{-1}(\tau)b(\tau) = [(T - \tau)\cos\varphi + \sin\varphi]\frac{1}{m - \mu\tau} \tag{2.6}$$

When $0 \leqslant \varphi \leqslant \pi - \operatorname{arctg} T$ (we assume that $t_0 = 0$) the function (2.6) is positive, and when $\pi \leqslant \varphi \leqslant 2\pi - \operatorname{arctg} T$ it is negative for all $\tau \in (0, T)$. Consequently, for any $T < m/\mu$ a complete range of values of φ exists (which means also a range of vectors $\eta \in S$), such that for any two different values of φ from that range, the identity (2.2) holds. Hence the boundary of the attainment region of system (2.4) for any $T < m/\mu$ has two corner points. They are obtained by substituting into formula (1.2) the functions

$$u(\tau) = \pm 1 \tag{2.7}$$

and only them. This result is well known in the literature $\mu = 0$.

Consider a second-order system with complex eigenvalues

$$x_1^{\cdot} = x_2, \ x_2^{\cdot} = -x_1 - 2\varepsilon x_2 + u \quad (|\varepsilon| < 1) \tag{2.8}$$

Taking into account the notation (2.5) when $\varepsilon = 0$ we obtain

$$\eta e^{A\xi}b = \sin(\xi + \varphi) \tag{2.9}$$

Let $t_0 = 0$ and $T < \pi$. Then for $0 \leqslant \varphi \leqslant \pi - T$ the function (2.9) is positive, and for $\pi \leqslant \varphi \leqslant 2\pi - T$ it is negative for all $\xi \in (0, T)$. Hence, when $T < \pi$ identity (2.3) holds for any two different values of φ from a certain range. When $T \geqslant \pi$ the identity (2.3) can obviously not hold for any vectors $\eta^{(1)}, \eta^{(2)}$ (and angles $\varphi^{(1)}, \varphi^{(2)}$). Thus, when $T < \pi$ the boundary of attainment of system (2.8) (in the case of $\varepsilon = 0$) has two corner points, and when $T \geqslant \pi$ it is everywhere smooth. This result can also be obtained by direct construction (analytically) of the region $Q(0, T)$. It can be shown that when $\varepsilon \neq 0$, the boundary $\Gamma(0, T)$ has two angular points when $T < \pi/\omega$, and when $T \geqslant \pi/\omega$ it is smooth $(\omega = \sqrt{1 - \varepsilon^2})$.

Definition 2. We will call the corner point of the boundary $\Gamma(t_0, T)$, which corresponds to the $(n - 1)$-dimensional subset of sphere S, the conical corner point.

This is natural, since the envelope of the supporting hypersurfaces of the region $Q(t_0, T)$ containing such a point is a conical surface. In second-order systems the corner points can obviously only be conical (Fig.1).

Under condition (1.7)

$$b(t) \neq 0 \quad (l_1(t) \neq 0) \tag{2.10}$$

for any $-\infty < t < \infty$. In a linear controllable system with constant coefficients, of course, $b \neq 0$.

Lemma 3. The conical corner points of the boundary $\Gamma(t_0, T)$ (if they exist) are obtained by substituting the functions (2.7), and only them, into (1.2).

$$x(T) = \pm \int_{t_0}^{T} \theta(T)\theta^{-1}(\tau) b(\tau)\, d\tau \tag{2.11}$$

We prove this by reductio ad absurdum. Let a conical corner point be obtained by using the control with switching at a certain $\tau \in (t_0, T)$. Then, by definition 2

$$\eta\theta(T)\theta^{-1}(\tau) b(\tau) = 0 \tag{2.12}$$

for all vectors η from some $(n-1)$-dimensional subset of the sphere S. But this is impossible, since the equality (2.1) isolates on an $(n-1)$-dimensional sphere S a manifold of dimensionality $(n-2)$ $(\theta(T)\theta^{-1}(\tau) b(\tau) \neq 0$, consequent on the nondegeneracy of the matrix $\theta(t)$ and inequality (2.10).

Lemma 3 implies that on the boundary $\Gamma(t_0, T)$ there are two conical oorner points or none (see system (2.8) and Theorem 3).

Since the function $\psi(\eta, T, \tau)$ depends continuously on the vector η, from Lemma 3 we have the following lemma.

Lemma 4. For conical points to exist on the boundary $\Gamma(t_0, T)$ it is necessary and sufficient that a vector η exists such that

$$\eta\theta(T)\theta^{-1}(\tau) b(\tau) > 0, \quad \forall \tau \in (t_0, T) \tag{2.13}$$

We will show that for any t_0 quantity T' exists such that for all $T \in (t_0, T')$ the set

$$S_{t_0, T} = \{\eta \in S \colon \eta\theta(T)\theta^{-1}(\tau) b(\tau) > 0, \quad \forall \tau \in (t_0, T)\}$$

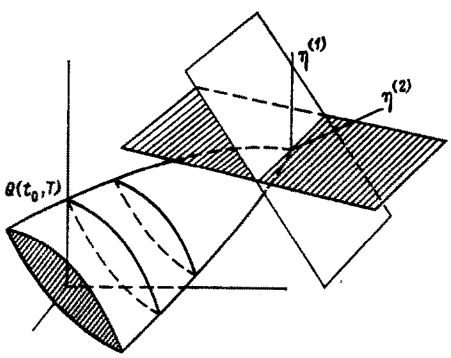

Fig.2

is nonempty and moreover $(n-1)$-dimensional. In fact, we select the vector η so that $\psi(\eta, t_0, t_0) = \eta b(t_0) > 0$ (we recall that $b(t_0) \neq 0$). The function $\psi(\eta, T, \tau)$ depends continuously on the arguments T and τ, a T' exists such that for all $T, \tau \in (t_0, T')$ the inequality (2.13) holds. Obviously, this inequality holds in some neighbourhood of the vector η on the $(n-1)$-dimensional sphere S. We denote by $-S_{t_0, T}$ the part of the sphere S that is symmetric to the set $S_{t_0, T}$ with respect to the origin of coordinates. The mapping (1.8) or (1.9) sets in correspondence to the sets $S_{t_0, T}$ and $-S_{t_0, T}$ two conical corner points of the boundary $\Gamma(t_0, T)$ which are symmetric with respect to the origin of coordinates (Figs.1 and 2; Fig.2 shows half of the attainment region in three-dimensional space).

Thus we have the following theorem.

Theorem 1. For any t_0 a T' exists such that the boundary $\Gamma(t_0, T)$ has exactly two conical corner points for every $T \in (t_0, T')$.

The fundamental matrix $e^{A\xi}$ of system (1.1) with constant coefficients may be written in the form /13/

$$e^{A\xi} = \sum_{k=1}^{r} \sum_{l=0}^{p_k-1} \alpha_{kl} e^{\lambda_k \xi} \xi^l$$

where λ_k $(k = 1, \ldots, r)$ are various eigenvalues of the matrix A of multiplicity p_k, and α_{kl} are constant matrices. Then

$$\eta e^{A\xi} b = \sum_{k=1}^{r} \sum_{l=0}^{p_k-1} \eta \alpha_{kl} b e^{\lambda_k \xi} \xi^l \tag{2.14}$$

Let us assume that among the numbers λ_k $(k = 1, \ldots, r)$ there is at least one real number, for instance λ_1. Of the n quantities $\eta\alpha_{kl}b$ we equate to zero $n-1$ of them

$$\eta\alpha_{kl}b = 0 \qquad (2.15)$$

$$\begin{pmatrix} l=1,\ldots,p_k-1; k=1 \\ l=0,1,\ldots,p_k-1; k=2,\ldots,r \end{pmatrix}$$

Only the quantity $\eta\alpha_{10}b$ remains. Under condition (1.6) Eqs.(2.15) are linearly independent and have as their solution only two unit vectors η^0 and $-\eta^0$. Substitutions of the vector η^0 into Eq.(2.14) gives

$$\eta^0 e^{A\xi}b = \eta^0\alpha_{10}be^{\lambda_0\xi} \qquad (2.16)$$

The function (2.16) retains its sign along the whole axis $-\infty < \xi < \infty$ (on substituting the vector $-\eta^0$ its sign changes). Hence using Lemma 4, we have the following theorem.

Theorem 2. If the matrices A and b of system (1.1) are constant and matrix A has at least one real eigenvalue, the boundary $\Gamma(t_0, T)$ has exactly two conical corner points for all t_0, T.

Let us assume now that all the eigenvalues of system (1.1) with constant coefficients are complex. Then for any vector $\eta \in S$ a quantity $T - t_0$ exists such that in the interval $(0, T - t_0)$ the function (2.14) has at least one zero. In turn, the vector η has on the sphere S a neighbourhood such that for each vector, the function (2.14) in the same interval has at least one zero. Selecting on the sphere S a finite covering we can ascertain the existence of a quantity ξ' such that in the interval $(0, \xi')$, function (2.14) has a least one zero for each vector $\eta \in S$. By Lemma 4 we can make the following statement.

Theorem 3. If the matrices A and b of system (1.1) are constant and matrix A has only complex eigenvalues, then a value of $\xi' > 0$ exists such that the boundary $\Gamma(t_0, T)$ has no conical corner points for $|T - t_0| > \xi'$.

Unlike second-order systems, the corner points of third-order and higher systems may not only be conical.

Under condition (1.7) a vector ζ^0 exists (hence for a given T also a vector $\eta^0 = \theta^{-1}(T)\zeta^0$), such that

$$\psi(\eta^0, T, t_0) = \zeta^0\theta^{-1}(t_0)\, l_1(t_0) = 0$$

$$\frac{d\psi(\eta^0, T, t_0)}{d\tau} = \zeta^0\theta^{-1}(t_0)\, l_2(t_0) > 0$$

The quantity T may be selected to be so close to t_0 that for the vector $\eta^0 = \theta^{-1}(T)\zeta^0$ the inequality $\psi(\eta^0, T, \tau) > 0$ holds for all $\tau \in (t_0, T)$. The vector $\eta^0 \in S_{t_0, T}$ lies on the boundary of the set $S_{t_0, T}$. It is clear that on the sphere S a neighbourhood of the point η^0 exists for whose every point η the function $\psi(\eta, T, \tau)$ has not more than one zero in the interval (t_0, T). Then for any τ fairly close to t_0, in that neighbourhood a vector $\eta^{(1)}$ can be found such that the function $\psi(\eta^{(1)}, T, \tau)$ in the interval (t_0, T) vanishes only at the point τ (see Eq.(2.12)). The function $\psi(\eta^{(1)}, T, \tau)$ in that zero of τ changes its sign from minus to plus. Equation (2.12) defines for fixed τ on the sphere S an $(n-2)$-dimensional set, which for $n \geqslant 3$ is nonempty. In that set a neighbourhood of the vector $\eta^{(1)}$ exists in which the function $\psi(\eta, T, \tau)$ in the interval (t_0, T) has a zero at the fixed point τ and only at that point. Obviously the control (1.5) corresponds to the vectors η from that neighbourhood with one instant τ of change of sign, and on the boundary $\Gamma(t_0, T)$ a corner point that is conical. Taking the values of τ from t_0 to $t_0 + \varepsilon$, where $|\varepsilon|$ is a fairly small number, we obtain a set of such corner points which can be called the edge /7/. The edge constructed above begins at the conical corner point (when $\tau = t_0$). Unlike the conical corner points which represent zero dimensional sets in space X, the edges are one-dimensional sets of corner points.

Definition 3. We shall call the corner point of the boundary $\Gamma(t_0, T)$ which corresponds to an $(n-2)$-dimensional subset of the sphere S, a point of one-dimensional edge.

The above reasoning contains the proof of the following theorem.

Theorem 4. For every t_0 a T' exists such that the boundary $\Gamma(t_0, T)$ has two conical corner points on the boundary $T \in (t_0, T')$, and from each such point at least one edge begins.

The following theorem can be proved.

Theorem 5. For any t_0 a T' exists such that for every $T \in (t_0, T')$ there are on the boundary $\Gamma(t_0, T)$ two conical corner points that are connected to one another by two edges.

The edges connect conical corner points when for each $\tau \in (t_0, T)$ a vector η exists such that the function $\psi(\eta, T, \tau) = 0$ at only one point of the interval (t_0, T). To obtain edges it is necessary to substitute into Eq.(1.2) all possible functions $u(t)$ that in one time interval have a value 1 and in another -1. The unique instant τ when the sign of these functions changes encompasses all values from the interval (t_0, T).

In third-order systems the singularities of the boundary $\Gamma(t_0, T)$ are exhausted by the presence of conical corner points and one-dimensional edges (Fig.2). When $|T - t_0|$ is small the attainment region in third-order systems is like a plum stone. This similarity is maintained for all $T - t_0$, if (1.1) is a stationary system and all three eigenvalues of the matrix A are real /7/.

In fourth-order systems the boundary $\Gamma(t_0, T)$ may have two-dimensional "edges", whose points correspond to the one-dimensional subset of the sphere S, and n-th order systems may have up to $(n-2)$-dimensional edges, whose points correspond to a one-dimensional subset of the sphere S.

Theorem 6. For any t_0 a T' exists such that the boundary $\Gamma(t_0, T)$ has for any $T \in (t_0, T')$ two conical corner points and, also, corner points that form one-, two-, three-, and ..., $(n-2)$-dimensional edges.

Let us prove this theorem. Under condition (1.7) a vector ζ° exists (for a given T this vector is used to determine the vector $\eta^\circ = \theta^{-1}(T)\zeta^\circ$) such that

$$\frac{d^i\psi(\eta^\circ, T, t_0)}{d\tau^i} = \zeta^\circ\theta^{-1}(t_0)\, l_{i+1}(t_0) = 0 \qquad (i = 0, 1, \ldots, k-1)$$

$$\frac{d^k\psi(\eta^\circ, T, t_0)}{d\tau^k} = \zeta^\circ\theta^{-1}(t_0)\, l_{k+1}(t_0) > 0$$

$$(1 \leqslant k \leqslant n-2; \quad \zeta^\circ = \theta(T)\eta^\circ)$$

The quantity T can be selected to be so close to t_0 that for the vector $\eta^\circ = \theta^{-1}(T)\zeta^\circ$ for all $\tau \in (t_0, T)$ the inequality $\psi(\eta^\circ, T, \tau) > 0$ holds. It can be ascertained that on the sphere S a neighbourhood of the vector η° exists, for each point of which the function $\psi(\eta, T, \tau)$ has in the interval (t_0, T) not more that k zeros. Then for any set of numbers $\tau_1, \ldots, \tau_k$ fairly close to t_0, a vector $\eta^{(1)}$ can be found in that neighbourhood such that the function $\psi(\eta^{(1)}, T, \tau)$ has in the interval (t_0, T) exactly k zeros of $\tau_1, \ldots, \tau_k$.

$$\eta^{(1)}\theta(T)\theta^{-1}(\tau_i)\, b(\tau_i) = 0 \quad (i = 1, \ldots, k) \tag{2.17}$$

Equation (2.17) defines for fixed $\tau_1, \ldots, \tau_k$ on the sphere S an $(n-k-1)$-dimensional set. In that set a neighbourhood of the vector $\eta^{(1)}$ exists where the function $\psi(\eta, T, \tau)$ vanishes in the interval (t_0, T) at fixed points $\tau_1, \ldots, \tau_k$ and only at these points. The control (1.5) obviously corresponds to the vectors in that neighbourhood, with switching instances $\tau_1, \ldots, \tau_k$, and at the boundary $\Gamma(t_0, T)$ a corner point corresponds to it. Taking the instants $\tau_1, \ldots, \tau_k$, we obtain a set of corner points that form a k-dimensional edge. The number k may have any value from unity to $n-2$. This proves Theorem 6, which generalizes Theorems 1 and 4.

Theorem 6 was proved in /7/ for all $T - t_0$ in the case when the matrices A and b are constant and all eigenvalues of the matrix A are real.

3. Extremal points of the boundary.

We introduce the following definition.

Definition 4. The point (vector) $x \in \Gamma(t_0, T)$ is called extremal, if on the boundary $\Gamma(t_0, T)$ it has a neighbourhood all the points of which distant from the origin of coordinates by not more (not less) than the point x.

For point $x \in \Gamma(t_0, T)$ to be extremal, the existence of a supporting hyperplane $\Pi(\eta)$ is necessary. That hyperplane passes through the point x such that

$$\eta = x^*/\|x\| \tag{3.1}$$

where $\|x\|$ is the norm of the vector x, and the asterisk denotes transposition. From Eqs.(1.8) and (3.1) it follows that the extremal vectors $x(T)$ satisfy the condition

$$x(T) = \int_{t_0}^{T} \theta(T)\theta^{-1}(\tau)\, b(\tau)\, \mathrm{sgn}\,[x^*(T)\,\theta(T)\theta^{-1}(\tau)\, b(\tau)\,(T - t_0)]\, d\tau \tag{3.2}$$

Equation (3.2) may be treated as follows: the extremal vector $x \in \Gamma(t_0, T)$ is an eigenvector of transformation (1.8) or (1.9) (apart from the normalizing multiplier).

If a conical corner point $x \in \Gamma(t_0, T)$, is extremal, it is distant from the origin of coordinates by not less than the neighbouring points; this follows from the convexity of the attainment region $Q(t_0, T)$. Hence we shall consider the conical corner point whose distance from the origin of coordinates is a maximum to be extremal. For the conical corner point $x \in \Gamma(t_0, T)$ to be the maximum distance from the origin of coordinates it is necessary and sufficient that it is possible to draw through that point x a supporting hyperplane orthogonal to the vector x.

Theorem 7. For the conical corner point of boundary $\Gamma(t_0, T)$ to be at the maximum distance from the origin of coordinates is necessary that

$$\operatorname{sgn}(T-t_0)\int_{t_0}^{T} b^*(\tau)(\theta^{-1}(\tau))^*\theta^*(T)\theta(T)\theta^{-1}(\xi)b(\xi)\,d\tau \geqslant 0$$
$$\forall\xi \in (t_0, T)$$

and sufficient that

$$\operatorname{sgn}(T-t_0)\int_{t_0}^{T} b^*(\tau)(\theta^{-1}(\tau))^*\theta^*(T)\theta(T)\theta^{-1}(\xi)b(\xi)\,d\tau > 0 \qquad (3.3)$$
$$\forall\xi \in (t_0, T)$$

To prove the theorem it is sufficient to recall Lemmas 3 (Eq.(2.11)) and 4, and Eq.(3.1).

According to Lemma 4, the strict inequality (3.3) is the sufficient condition for conical corner points to be present. Hence by Theorem 7 the strict inequality (3.3) is the sufficient condition for conical corner points that are at the maximum distance from the origin of coordinates to exist on the boundary $\Gamma(t_0, T)$.

For constant matrices A and b, inequality (3.3) takes the form

$$\operatorname{sgn}(T-t_0)\int_{0}^{T-t_0} b^* e^{A^*\tau}e^{A\xi}b\,d\tau > 0, \quad \forall\xi \in (0, T-t_0)$$

The integrand function in (3.3) (let us denote it by $W(\tau, \xi)$) continuously depends on both of its arguments. Under condition (2.10) $W(\tau, \tau) > 0$ for all $-\infty < \tau < \infty$. Hence taking into account Theorem 1, we obtain the following statement.

Theorem 8. For every t_0 a T' exists such that the boundary $\Gamma(t_0, T)$ for every $T \in (t_0, T')$ has two conical corner points that are at the maximum distance from the origin of coordinates.

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